

Reproduction numbers and the expanding fronts for a diffusion-advection SIS model in heterogeneous time-periodic environment*

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Abstract. This paper deals with a simplified SIS model, which describes the transmission of the disease in time-periodic heterogeneous environment. To understand the impact of spatial heterogeneity of environment and small advection on the persistence and eradication of an infectious disease, the left and right free boundaries are introduced to represent the expanding fronts. The basic reproduction numbers R_0^D and $R_0^F(t)$, which depends on spatial heterogeneity, temporal periodicity and advection, is introduced. A spreading-vanishing dichotomy is established and sufficient conditions for the spreading and vanishing of the disease are given. The asymptotic spreading speeds for the left and right fronts are also presented.

MSC: primary: 35R35; secondary: 35K60

Keywords: Reaction-diffusion systems; advection; diffusive SIS model; time-periodic; basic reproduction number;

1 Introduction

To understand the transmission of infectious diseases, many mathematical models have been made and investigated. Considering spatial diffusion and environmental heterogeneity, Allen, Bolker, Lou and Nevai in [1] proposed an SIS epidemic reaction-diffusion model

$$\begin{cases} S_t - d_S \Delta S = -\frac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega, \ t > 0, \\ I_t - d_I \Delta I = \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega, \ t > 0, \\ \frac{\partial S}{\partial \eta} = \frac{\partial I}{\partial \eta} = 0, & x \in \partial\Omega, \ t > 0, \end{cases} \quad (1.1)$$

*The work is partially supported by the NSFC of China (Grant No. 11371311 and 11501494), the High-End Talent Plan of Yangzhou University.

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where $S(x, t)$ and $I(x, t)$ represent the density of susceptible and infected individuals at location x and time t , respectively, the positive constants d_S and d_I denote the corresponding diffusion rates for the susceptible and infected populations, $\beta(x)$ and $\gamma(x)$ are positive Hölder continuous functions, which account for spatial dependent rates of disease contact transmission and disease recovery at x , respectively. The term $\frac{\beta(x)SI}{S+I}$ is the standard incidence of disease.

As in [1], we say that x is a **high-risk site** if the local disease transmission rate $\beta(x)$ is greater than the local disease recovery rate $\gamma(x)$. An **low-risk site** is defined in a similar manner. The habitat Ω is characterized as **high-risk** (or **low-risk**) if the spatial average $(\frac{1}{|\Omega|} \int_{\Omega} \beta(x) dx)$ of the transmission rate is greater than (or less than) the spatial average $(\frac{1}{|\Omega|} \int_{\Omega} \gamma(x) dx)$ of the recovery rate, respectively.

In some recent work [10, 12, 13], Peng et al. further investigated the asymptotic behavior and global stability of the endemic equilibrium for system (1.1) subject to the Neumann boundary conditions, and provided much understanding of the impacts of large and small diffusion rates of the susceptible and infected population on the persistence and extinction of the disease.

To focus on the new phenomena induced by spatial heterogeneity of environment, we assume that the population $N(x, t)$ is constant in space for all time, that is, $N(x, t) \equiv N^*$ for $x \in \Omega$ and $t \geq 0$ and consider the corresponding free boundary problem

$$\begin{cases} I_t - d_I I_{xx} + \alpha I_x = (\beta(x, t) - \gamma(x, t))I - \frac{\beta(x, t)}{N^*} I^2, & g(t) < x < h(t), \quad t > 0, \\ I(g(t), t) = 0, \quad g'(t) = -\mu I_x(g(t), t), & t > 0, \\ I(h(t), t) = 0, \quad h'(t) = -\mu I_x(h(t), t), & t > 0, \\ g(0) = -h_0, \quad h(0) = h_0, \quad I(x, 0) = I_0(x), & -h_0 \leq x \leq h_0, \end{cases} \quad (1.2)$$

where $x = g(t)$ and $x = h(t)$ are the moving left and right boundaries to be defined, h_0 , d_I , α and μ are positive constants. α and μ are referred as the advection rate and the expanding capability, respectively. $\beta(x, t), \gamma(x, t) \in C^{\nu_0, \frac{\nu_0}{2}}(\mathbb{R} \times [0, \infty))$ for some $\nu_0 \in (0, 1)$, which account for spatial dependent rates of disease contact transmission and disease recovery, respectively. We assume that $\beta(x, t)$ and $\gamma(x, t)$ are positive and bounded, that is, there exist positive constants $\beta_1, \beta_2, \gamma_1$ and γ_2 such that $\beta_1 \leq \beta(x, t) \leq \beta_2$ and $\gamma_1 \leq \gamma(x, t) \leq \gamma_2$ in $\mathbb{R} \times [0, \infty)$. Considering periodic environment, we assume that $\beta(x, t), \gamma(x, t)$ are periodic in t with the same period T (i.e., $\beta(x, t+T) = \beta(x, t)$, $\gamma(x, t+T) = \gamma(x, t)$ for all $t \in \mathbb{R}$). Further, in the paper we assume

$$(H_1) \quad \lim_{x \rightarrow \pm\infty} \beta(x, t) = \beta_{\infty}(t), \quad \lim_{x \rightarrow \pm\infty} \gamma(x, t) = \gamma_{\infty}(t) \text{ uniformly for } t \in [0, T],$$

which means that far sites of the habitat are similar.

In this paper, we only consider the small advection and assume that

$$(H_2) \quad \alpha < 2\sqrt{d_I \left[\frac{1}{T} \int_0^T (\beta_{\infty}(t) - \gamma_{\infty}(t)) dt \right]}.$$

The initial distribution of the infected populations $I_0(x)$ is nonnegative and satisfies

$$I_0 \in C^2[-h_0, h_0], \quad I_0(-h_0) = I_0(h_0) = 0 \text{ and } 0 < I_0(x) \leq N^*, \quad x \in (-h_0, h_0), \quad (1.3)$$

where the condition (1.3) indicates that at the beginning, the infected exists in the area with $x \in (-h_0, h_0)$, but for the area $|x| \geq h_0$, no infected happens yet. Therefore, the model means that beyond the left boundary $x = g(t)$ and the right boundary $x = h(t)$, there is only susceptible, no infected individuals.

The equations governing the free boundary, the spreading front, $h'(t) = -\mu I_x(h(t), t)$ and $g'(t) = -\mu I_x(g(t), t)$, are the special cases of the well-known Stefan condition, which has been established in [9] for the diffusive populations. The positive constant μ measures the ability of the infected transmit and diffuse towards the new area.

After we finished the first version of this paper, we found the papers ([2, 15]) dealing with similar problem, which describing spatial spreading of the species without advection. In our paper, we considered the effect of advection by using the basic reproduction numbers. We therefore emphasized the different consideration and omitted some similar proofs.

This rest of the paper is arranged as follows. In the next section, the global existence and uniqueness of the solution to (1.2) are presented by using a contraction mapping theorem, comparison principle is also employed. Section 3 is devoted to introducing the basic reproduction numbers and deriving their analytical properties. Sufficient conditions for the disease to vanish or spread are given in section 4. The asymptotic spreading speeds are also presented.

2 Preliminaries

In this section, we first present some fundamental results on solutions of problem (1.2), we omit the proof since it is standard, see also Lemma 2.2, Theorems 2.1 and 2.2 in [6].

Theorem 2.1 *For any given I_0 satisfying (1.3), and any $\nu \in (0, 1)$, problem (1.2) admits a unique global solution*

$$(I; g, h) \in C^{1+\nu, (1+\nu)/2}([g(t), h(t)] \times [0, +\infty)) \times C^{1+\nu/2}([0, T]) \times C^{1+\nu/2}([0, +\infty));$$

moreover,

$$\begin{aligned} 0 < I(x, t) \leq N^* \quad \text{for } g(t) < x < h(t), \quad t \in (0, T_0]. \\ -C_1 \leq g'(t) < 0 \text{ and } 0 < h'(t) \leq C_1 \quad t \in (0, T_0]. \end{aligned}$$

for some constants C_1 and T_0 .

For later applications, we exhibit the comparison principle, which is similar to Lemma 3.5 in [5].

Lemma 2.2 *(The Comparison Principle) Assume that $T \in (0, \infty)$, $\bar{g}, \bar{h}, \underline{g}, \underline{h} \in C^1([0, T])$, $\bar{I}(x, t) \in C([\bar{g}(t), \bar{h}(t)] \times [0, T]) \cap C^{2,1}((\bar{g}(t), \bar{h}(t)) \times (0, T))$, $\underline{I}(x, t) \in C([\underline{g}(t), \underline{h}(t)] \times [0, T]) \cap$*

$C^{2,1}((\underline{g}(t), \underline{h}(t)) \times (0, T])$, and

$$\left\{ \begin{array}{ll} \bar{I}_t - d_I \bar{I}_{xx} + \alpha \bar{I}_x \geq (\beta(x, t) - \gamma(x, t)) \bar{I} - \frac{\beta(x, t)}{N^*} \bar{I}^2, & \bar{g}(t) < x < \bar{h}(t), \quad 0 < t \leq T, \\ \underline{I}_t - d_I \underline{I}_{xx} + \alpha \underline{I}_x \leq (\beta(x, t) - \gamma(x, t)) \underline{I} - \frac{\beta(x, t)}{N^*} \underline{I}^2, & \underline{g}(t) < x < \underline{h}(t), \quad 0 < t \leq T, \\ \bar{I}(\bar{g}(t), t) = 0, \quad \bar{g}'(t) \leq -\mu \bar{I}_x(\bar{g}(t), t), & 0 < t \leq T, \\ \underline{I}(\underline{g}(t), t) = 0, \quad \underline{g}'(t) \geq -\mu \underline{I}_x(\underline{g}(t), t), & 0 < t \leq T, \\ \bar{I}(\bar{h}(t), t) = 0, \quad \bar{h}'(t) \geq -\mu \bar{I}_x(\bar{h}(t), t), & 0 < t \leq T, \\ \underline{I}(\underline{h}(t), t) = 0, \quad \underline{h}'(t) \leq -\mu \underline{I}_x(\underline{h}(t), t), & 0 < t \leq T, \\ \bar{g}(0) \leq -h_0 < h_0 \leq \bar{h}(0), \quad I_0(x) \leq \bar{I}(x, 0), & -h_0 \leq x \leq h_0, \\ -h_0 \leq \underline{g}(0) \leq \underline{h}(0) \leq h_0, \quad \underline{I}(x, 0) \leq I_0(x), & \underline{g}(0) \leq x \leq \underline{h}(0), \end{array} \right.$$

then the solution $(I(x, t); g(t), h(t))$ to the free boundary problem (1.2) satisfies

$$\bar{g}(t) \leq g(t) \leq \underline{g}(t), \quad \underline{h}(t) \leq h(t) \leq \bar{h}(t) \text{ for } t \in [0, T],$$

$$\underline{I}(x, t) \leq I(x, t) \text{ for } (x, t) \in [\underline{g}(t), \underline{h}(t)] \times [0, T],$$

$$I(x, t) \leq \bar{I}(x, t) \text{ for } (x, t) \in [g(t), h(t)] \times [0, T].$$

The pair $(\bar{u}; \bar{g}, \bar{h})$ in Lemma 2.2 is usually called an upper solution of the problem (1.2) and $(\underline{u}; \underline{g}, \underline{h})$ is then called a lower solution. To examine the dependence of the solution on the expanding capability μ , we write the solution as $(I^\mu; g^\mu, h^\mu)$. As a corollary of Lemma 2.2, we have the following monotonicity:

Corollary 2.3 *For fixed $I_0, \alpha, h_0, \beta(x, t)$ and $\gamma(x, t)$. If $\mu_1 \leq \mu_2$. Then $I^{\mu_1}(x, t) \leq I^{\mu_2}(x, t)$ in $[g^{\mu_1}(t), h^{\mu_1}(t)] \times (0, \infty)$ and $g^{\mu_1}(t) \leq g^{\mu_2}(t)$, $h^{\mu_1}(t) \leq h^{\mu_2}(t)$ in $(0, \infty)$.*

3 Basic reproduction numbers

In this section, we first present the basic reproduction number and its properties for the corresponding system in a fixed interval, and then define the basic reproduction number for the free boundary problem (1.2). The basic reproduction numbers are related to the eigenvalues of corresponding periodic-parabolic eigenvalue problems.

Consider the reaction-diffusion-advection problem

$$\left\{ \begin{array}{ll} I_t - d_I I_{xx} + \alpha I_x = -\gamma(x, t) I, & x \in (h_1, h_2), \quad t > 0, \\ \phi(h_1, t) = \phi(h_2, t) = 0, & t > 0, \end{array} \right. \quad (3.1)$$

Let $U(t, s)$ be the evolution operator of (3.1), then there exist constants $C, \omega > 0$ such that $\|U(t, s)\| \leq C e^{-\omega(t-s)}$ for any $t \geq s, t, s \in \mathbb{R}$. As in [13], we introduce the next generation operator

$$L(\psi)(t) := \int_0^\infty U(t, t-s) \beta(\cdot, t-s) \psi(\cdot, t-s) ds,$$

where $\psi \in C_T := \{u : t \rightarrow u(t) \in C[h_1, h_2], u(0)(x) = u(T)(x)\}$. From the definition, we know that L is continuous, strongly positive and compact on C_T . We now define the basic reproduction number of system (1.2) as the spectral radius of L , that is,

$$R_0^D = R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2]) = \rho(L).$$

With the above definition, we have the following equivalent characterizations.

Lemma 3.1 (i) $R_0^D = \mu_0$, where μ_0 is the unique principal eigenvalue of periodic-parabolic eigenvalue problem

$$\begin{cases} \phi_t - d_I \phi_{xx} + \alpha \phi_x = -\gamma(x, t)\phi + \frac{\beta(x, t)}{\mu_0} \phi, & x \in (h_1, h_2), t > 0, \\ \phi(h_1, t) = \phi(h_2, t) = 0, & t > 0, \\ \phi(x, 0) = \phi(x, T), & x \in [h_1, h_2]. \end{cases} \quad (3.2)$$

(ii) $1 - R_0^D$ has the same sign as λ_0 , where λ_0 is the principal eigenvalue of periodic-parabolic eigenvalue problem

$$\begin{cases} \psi_t - d_I \psi_{xx} + \alpha \psi_x = \beta(x, t)\psi - \gamma(x, t)\psi + \lambda_0 \psi, & x \in (h_1, h_2), t > 0, \\ \psi(h_1, t) = \psi(h_2, t) = 0, & t > 0, \\ \psi(x, 0) = \psi(x, T), & x \in [h_1, h_2]. \end{cases} \quad (3.3)$$

The proof of this lemma is similar as that of Lemmas 2.1 and 2.2 in [13]. The existence of the unique principal eigenvalue μ_0 of (3.2) and λ_0 of (3.3) can be seen from Section 16 (Theorem 16.1) and Section 14 in [8], respectively. Moreover, the eigenfunction $\phi(x, t)$ of (3.2) corresponding to μ_0 and the eigenfunction $\psi(x, t)$ of (3.3) corresponding to λ_0 are positive in $(h_1, h_2) \times \mathbb{R}$.

To see the properties of the basic reproduction number R_0^D , let us see the two special cases. In the first case, if $\beta(x, t)$ and $\gamma(x, t)$ are spatially homogeneous, we have:

Theorem 3.2 If $\beta(x, t) \equiv \beta(t)$ and $\gamma(x, t) \equiv \gamma(t)$, then

- (i) $R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2]) = \frac{\frac{1}{T} \int_0^T \beta(t) dt}{d_I (\frac{\pi}{h_2 - h_1})^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma(t) dt}$;
- (ii) $R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2])$ is monotone increasing with respect to $\beta(t)$, and decreasing with respect to α and $\gamma(t)$;
- (iii) $R_0^D(d_I, \alpha, \beta(t), \gamma(t), \cdot)$ is an increasing function for fixed $d_I, \alpha, \beta(t), \gamma(t)$, in the sense that $R_0^D(d_I, \alpha, \beta(t), \gamma(t), I_1) \leq R_0^D(d_I, \alpha, \beta(t), \gamma(t), I_2)$ provided that $I_1 \subseteq I_2 \subseteq \mathbb{R}^1$;
- (iv)

$$\lim_{(h_2 - h_1) \rightarrow 0^+} R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2]) = 0,$$

and

$$\lim_{(h_2 - h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2]) = \frac{\frac{1}{T} \int_0^T \beta(t) dt}{\frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma(t) dt}.$$

Moreover, there exists a positive constants h^* such that $R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_1 + h^*]) = 1$, $R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_1 + h]) > 1$ for $h > h^*$ and $R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_1 + h]) < 1$ for $h < h^*$ provided that (H_1) holds.

Proof: Let

$$\mu^* = \frac{\frac{1}{T} \int_0^T \beta(t) dt}{d_I \left(\frac{\pi}{h_2 - h_1} \right)^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma(t) dt},$$

$$u(x, t) = e^{\int_0^t (-\gamma(s) - d_I \left(\frac{\pi}{h_2 - h_1} \right)^2 - \frac{\alpha^2}{4d_I} + \frac{1}{\mu^*} \beta(s)) ds} e^{\frac{\alpha x}{2d_I}} \cos \frac{\pi}{h_2 - h_1} \left(x - \frac{h_2 - h_1}{2} \right),$$

it is easy to see that $u(x, t) = u(x, t + T)$ and $u(x, t)$ is a positive T -periodic solution to (3.2) with $\mu_0 = \mu^*$. The result of (i) is follows from the uniqueness of the principal eigenvalue of (3.2), and the conclusions of (ii) – (iv) follow from the expression of $R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2])$ in (i) directly. \square

For the other special case, $\beta(x, t) \equiv \beta(x)$ and $\gamma(x, t) \equiv \gamma(x)$, we have

Theorem 3.3 *The following assertions hold.*

- (a) $R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_2]) = \sup_{\psi \in H_0^1(h_1, h_2), \psi \neq 0} \left\{ \frac{\int_{h_1}^{h_2} \beta \psi^2 dx}{\int_{h_1}^{h_2} (d_I \psi_x^2 + \frac{\alpha^2}{4d_I} \psi^2 + \gamma \psi^2) dx} \right\}$ is a positive and monotone decreasing function of α ;
- (b) If $I_1 \subseteq I_2 \subseteq \mathbb{R}^1$, then $R_0^D(d_I, \alpha, \beta(x), \gamma(x), I_1) \leq R_0^D(d_I, \alpha, \beta(x), \gamma(x), I_2)$, with strict inequality if $I_2 \setminus I_1$ is a nonempty open set.
- (c) If $\beta(x, t) \equiv \beta_\infty$ and $\gamma(x, t) \equiv \gamma_\infty$, then

$$R_0^D(d_I, \alpha, \beta_\infty, \gamma_\infty, [h_1, h_2]) = \frac{\beta_\infty}{d_I \left(\frac{\pi}{h_2 - h_1} \right)^2 + \frac{\alpha^2}{4d_I} + \gamma_\infty}.$$

- (d) If (H_1) holds, then

$$\lim_{(h_2 - h_1) \rightarrow 0^+} R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_2]) = 0,$$

and

$$\lim_{(h_2 - h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_2]) \geq \frac{\beta_\infty}{\frac{\alpha^2}{4d_I} + \gamma_\infty}.$$

Furthermore, if the assumption (H_2) holds, we can find a positive constant h^* such that $R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_1 + h^*]) = 1$ and $R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_1 + h]) > 1$ for $h > h^*$, $R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_1 + h]) < 1$ for $h < h^*$.

Proof: Let μ_0 be the principal eigenvalue of the elliptic problem

$$\begin{cases} -d_I \phi_{xx} + \alpha \phi_x = -\gamma(x) \phi + \frac{\beta(x)}{\mu_0} \phi, & x \in (h_1, h_2), \\ \phi(h_1) = \phi(h_2) = 0. \end{cases} \quad (3.4)$$

It follows from Lemma 3.1 (i) and the variational methods that

$$R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_2]) = \mu_0 = \sup_{\phi \in H_0^1(h_1, h_2), \phi \neq 0} \left\{ \frac{\int_{h_1}^{h_2} \beta e^{-\alpha x/d_I} \phi^2 dx}{\int_{h_1}^{h_2} (d_I e^{-\alpha x/d_I} \phi_x^2 + \gamma e^{-\alpha x/d_I} \phi^2) dx} \right\}.$$

If $\phi \in H_0^1(h_1, h_2)$, then $\psi = e^{-\alpha x/(2d_I)}\phi \in H_0^1(h_1, h_2)$, therefore taking $\phi = e^{\alpha x/(2d_I)}\psi$ gives that

$$R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_2]) = \sup_{\psi \in H_0^1(h_1, h_2), \psi \neq 0} \left\{ \frac{\int_{h_1}^{h_2} \beta \psi^2 dx}{\int_{h_1}^{h_2} (d_I \psi_x^2 + \frac{\alpha^2}{4d_I} \psi^2 + \gamma \psi^2) dx} \right\}.$$

Then (a)-(c) hold directly from the formulation of $R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_2])$.

We then verify (d). It's easy to see that

$$\lim_{(h_2-h_1) \rightarrow 0^+} R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2]) = 0.$$

Assertion $\lim_{(h_2-h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(t), \gamma(t), [h_1, h_2]) \geq \frac{\beta_\infty}{\frac{\alpha^2}{4d_I} + \gamma_\infty}$ follows from the proof of Theorem 3.2 (e) [6] by little modification. In view of (b) and (H_2) , we can find a constant $h^* > 0$ such that $R_0^D([h_1, h_1 + h^*]) := R_0^D(d_I, \alpha, \beta(x), \gamma(x), [h_1, h_1 + h^*]) = 1$ and

$$R_0^D([h_1, h_1 + h]) > 1 \text{ for } h > h^*, R_0^D([h_1, h_1 + h]) < 1 \text{ for } h < h^*.$$

□

With the above properties in mind, we give some properties of $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$. Firstly, we present some monotonicity.

Theorem 3.4 *The following assertions hold.*

(1) $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$ is a monotone increasing with respect to $\beta(x, t)$ and decreasing with respect to $\gamma(x, t)$, that is,

$$R_0^D(d_I, \alpha, \beta_*(x, t), \gamma^*(x, t), [h_1, h_2]) \leq R_0^D(d_I, \alpha, \beta^*(x, t), \gamma_*(x, t), [h_1, h_2])$$

if $\beta_*(x, t) \leq \beta^*(x, t)$ and $\gamma_*(x, t) \leq \gamma^*(x, t)$;

(2) $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$ is a positive bounded function and satisfies

$$\frac{\frac{1}{T} \int_0^T \beta_m(t) dt}{d_I(\frac{\pi}{h_2-h_1})^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_M(t) dt} \leq R_0^D \leq \frac{\frac{1}{T} \int_0^T \beta_M(t) dt}{d_I(\frac{\pi}{h_2-h_1})^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_m(t) dt},$$

where $\gamma_m(t) = \min_{x \in [h_1, h_2]} \gamma(x, t)$, $\gamma_M(t) = \max_{x \in [h_1, h_2]} \gamma(x, t)$ and $\beta_m(t), \beta_M(t)$ are defined similarly;

(3) $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), \cdot)$ is an increasing function for fixed $d_I, \alpha, \beta(x, t), \gamma(x, t)$, in the sense that $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), I_1) \leq R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), I_2)$ provided that $I_1 \subseteq I_2 \subseteq \mathbb{R}^1$;

(4) Assume that (H_1) holds, then

$$\lim_{(h_2-h_1) \rightarrow 0^+} R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2]) = 0,$$

$$\liminf_{(h_2-h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2]) \geq \frac{\frac{1}{T} \int_0^T \beta_\infty(t) dt}{\frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_\infty(t) dt}.$$

Furthermore, there exists a unique positive constant h^* such that

$$R_0^D(h^*) := R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_1 + h^*]) = 1$$

and $R_0^D([h_1, h_1 + h]) > 1$ for $h > h^*$, $R_0^D([h_1, h_1 +]) < 1$ for $h < h^*$ provided that (H_2) holds.

Proof: (1) For any $\beta_1(x, t) \leq \beta_2(x, t)$, denote $\mu_0^i (i = 1, 2)$ is the principal eigenvalue of (3.2) and the corresponding eigenfunction is $\phi^i > 0 (i = 1, 2)$ in (h_1, h_2) . It's well-known (Theorem 7.2 in [8]) that μ_0^2 is also the eigenvalue of

$$\begin{cases} -\psi_t - d_I \psi_{xx} - \alpha \psi_x = -\gamma(x, t) \phi + \frac{\beta_2(x, t)}{\mu_0^2} \psi, & x \in (h_1, h_2), t > 0, \\ \psi(h_1, t) = \psi(h_2, t) = 0, & t > 0, \\ \psi(x, 0) = \psi(x, T), & x \in [h_1, h_2]. \end{cases} \quad (3.5)$$

It's corresponding eigenfunction is denoted by $\psi_2 > 0$ in $[h_1, h_2]$.

Next we apply the multiply-multiply-subtract-integrate trick. Multiply the equation of ϕ_1 by ψ_2 and the equation of ψ_2 by ϕ_1 . Then subtract the two equations and integrate over $(h_1, h_2) \times (0, T)$ obtain

$$\frac{\int_0^T \int_{h_1}^{h_2} \beta_1(x, t) \phi_1 \psi_2 dx dt}{\mu_0^1} = \frac{\int_0^T \int_{h_1}^{h_2} \beta_2(x, t) \phi_1 \psi_2 dx dt}{\mu_0^2}.$$

Hence, $\mu_0^1 \leq \mu_0^2$. From Lemma 3.1 (i), it follows that

$$R_0^D(d_I, \alpha, \beta_1(x, t), \gamma(x, t), [h_1, h_2]) \leq R_0^D(d_I, \alpha, \beta_2(x, t), \gamma(x, t), [h_1, h_2])$$

Therefore, $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$ is an increasing function of $\beta(x, t)$. Similarly, we deduce that $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$ is decreasing of $\gamma(x, t)$.

(2) We first get an upper bound of R_0^D by considering the eigenvalue problem

$$\begin{cases} \tilde{\phi}_t - d_I \tilde{\phi}_{xx} + \alpha \tilde{\phi} = -\gamma_m(t) \tilde{\phi} + \frac{\beta_M(t)}{\tilde{\mu}} \tilde{\phi}, & x \in (h_1, h_2), t > 0, \\ \tilde{\phi}(h_1, t) = \tilde{\phi}(h_2, t) = 0, & t > 0, \\ \tilde{\phi}(x, 0) = \tilde{\phi}(x, T), & x \in [h_1, h_2]. \end{cases} \quad (3.6)$$

It follows from Theorem 3.2 that

$$\tilde{\mu} = \frac{\frac{1}{T} \int_0^T \beta_M(t) dt}{d_I \left(\frac{\pi}{h_2 - h_1} \right)^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_m(t) dt}.$$

We apply the monotonicity of $\beta(x, t)$, $\gamma(x, t)$ in (1) to deduce that

$$R_0^D(d_I, \alpha, \beta_1(x, t), \gamma(x, t), [h_1, h_2]) \leq \tilde{\mu}.$$

Similarly, we obtain

$$R_0^D(d_I, \alpha, \beta_1(x, t), \gamma(x, t), [h_1, h_2]) \geq \frac{\frac{1}{T} \int_0^T \beta_m(t) dt}{d_I \left(\frac{\pi}{h_2 - h_1} \right)^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_M(t) dt}.$$

(3) Without loss of generality, we assume that $I_1 = [0, l_1]$, $I_2 = [0, l_2]$ and $0 < l_1 \leq l_2$. We will show that $R_0^D(I_1) \leq R_0^D(I_2)$. Denote ϕ be the eigenfunction corresponding to $R_0^D(I_1)$. Then we know that $R_0^D(I_2)$ is the principal eigenvalue of (3.5) and the corresponding eigenfunction is $\psi(x, t) (> 0)$.

Multiply the equation of ψ by ϕ and the equation of ϕ by ψ , then integrate them over $(0, l_1) \times (0, T)$ and subtract the resulting identities to obtain

$$\left(\frac{1}{R_0^D(I_1)} - \frac{1}{R_0^D(I_2)} \right) \int_0^T \int_0^{l_1} \beta(x, t) \phi \psi dx dt = -d_I \int_0^T \psi(l_1, t) \phi_x(l_1, t) dt.$$

Applying the Hopf Lemma to the equation of ϕ in $(0, l_1) \times (0, T)$ and we get $\phi_x(l_1, t) < 0$. It follows that $\left(\frac{1}{R_0^D(I_1)} - \frac{1}{R_0^D(I_2)} \right) \int_0^T \int_0^{l_1} \beta(x, t) \phi \psi dx dt > 0$. Therefore, $R_0^D(I_1) \leq R_0^D(I_2)$.

Finally, we verify (4). Combine with the properties of $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$ in (2), we easily deduce that

$$\lim_{(h_2 - h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2]) = 0.$$

Now, we show that

$$\liminf_{(h_2 - h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2]) \geq \frac{\frac{1}{T} \int_0^T \beta_\infty(t) dt}{\frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_\infty(t) dt}.$$

It follows from the assumption of (H_1) that for any $\varepsilon > 0$, there exists a positive constant L_0 such that for $|x| \geq L_0$,

$$|\beta(x, t) - \beta_\infty| < \varepsilon, \quad |\gamma(x, t) - \gamma_\infty| < \varepsilon. \quad (3.7)$$

Without loss of generality, we assume that $h_1 = 0$ and $h_2 \rightarrow \infty$. For any $L \geq 2L_0$, using the monotonicity of R_0^D with respect to β, γ and interval $[h_1, h_2]$ gives

$$\begin{aligned} & R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [0, L]) \\ & \geq R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [L/2, L]) \\ & \geq R_0^D(d_I, \alpha, \beta_\infty(t) - \varepsilon, \gamma_\infty(t) + \varepsilon, [L/2, L]) \quad (\text{by (3.7)}) \\ & = \frac{\frac{1}{T} \int_0^T (\beta_\infty(t) - \varepsilon) dt}{d_I \left(\frac{\pi}{L - L/2} \right)^2 + \frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T (\gamma_\infty(t) + \varepsilon) dt} \quad (\text{by Theorem 3.2}), \end{aligned}$$

which together with the arbitrariness of small ε gives

$$\liminf_{h_2 \rightarrow +\infty} R_0^D([0, h_2]) \geq \liminf_{L \rightarrow \infty} R_0^D([0, L]) \geq \frac{\frac{1}{T} \int_0^T \beta_\infty(t) dt}{\frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_\infty(t) dt}.$$

In view of the assumption of (H_2) holds, then we know that

$$\liminf_{(h_2 - h_1) \rightarrow +\infty} R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2]) \geq 1.$$

As $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_2])$ is increasing with $(h_2 - h_1)$, we can find $h^* > 0$ such that $R_0^D([h_1, h_1 + h^*]) := R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [h_1, h_1 + h^*]) = 1$ and

$$R_0^D([h_1, h_1 + h]) > 1 \text{ for } h > h^*, R_0^D([h_1, h_1 + h]) < 1 \text{ for } h < h^*.$$

□

Noticing that the interval $[g(\tau), h(\tau)]$, where the solution for the free boundary problem (1.2) exist, is changing with τ , so the basic reproduction number is not a constant and should be changing. Now we introduced the basic reproduction number $R_0^F(\tau)$ for the free boundary problem (1.2) by

$$R_0^F(\tau) := R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [g(\tau), h(\tau)]),$$

it follows from Theorems 2.1 and 3.4 that

Theorem 3.5 $R_0^F(\tau)$ is strictly monotone increasing function of τ , that is if $\tau_1 < \tau_2$, then $R_0^F(\tau_1) < R_0^F(\tau_2)$. Moreover, if (H_1) holds and $h(\tau) - g(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$, then $\lim_{\tau \rightarrow \infty} R_0^F(\tau) \geq \frac{\frac{1}{T} \int_0^T (\beta_\infty(t) dt)}{\frac{\alpha^2}{4d_I} + \frac{1}{T} \int_0^T \gamma_\infty(t) dt}$.

Remark 3.1 Assume that (H_1) and (H_2) hold. By Theorem 3.5, we have $R_0^F(\tau_0) > 1$ for some $\tau_0 > 0$ provided that $h(\tau) - g(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$.

4 Spreading-vanishing

It follows from Theorem 2.1 that $x = g(t)$ is monotonic decreasing and $x = h(t)$ is monotonic increasing, so there exist $g_\infty \in [-\infty, -h_0]$ and $h_\infty \in (h_0, +\infty]$ such that $\lim_{t \rightarrow +\infty} g(t) = g_\infty$ and $\lim_{t \rightarrow +\infty} h(t) = h_\infty$. The following spreading-vanishing dichotomy has been given in [2, 15] for the free boundary problem in time-periodic environment without advection.

Lemma 4.1 Assume that (H_1) and (H_2) hold. Then, the following alternative holds: Either (i) vanishing: $-\infty < g_\infty < h_\infty < \infty$, and

$$R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [g_\infty, h_\infty]) \leq 1 \text{ and } \lim_{t \rightarrow \infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0;$$

or (ii) spreading: $-g_\infty = \infty = h_\infty$, then

$$\lim_{n \rightarrow \infty} I(x, t + nT) = \hat{U}(x, t) \text{ locally uniformly for } (x, t) \in [0, \infty) \times [0, T],$$

where $\hat{U}(x, t)$ is the unique positive T -periodic solution of the problem

$$\begin{cases} U_t - d_I U_{xx} + \alpha U_x = (\beta(x, t) - \gamma(x, t))U - \frac{\beta(x, t)}{N^*} U^2, & x \in \mathbb{R}, 0 \leq t \leq T, \\ U(x, 0) = U(x, T), & x \in \mathbb{R}. \end{cases} \quad (4.1)$$

Proof: We first show that both h_∞ and g_∞ are finite or infinite simultaneously. In fact, if $h_\infty < \infty$, we can prove that $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [g_\infty, h_\infty]) \leq 1$ as in Lemma 4.1 in [6] by contradiction, which together with the assumption, implies that $g_\infty > -\infty$ by Theorem 3.5.

In the case $-\infty < g_\infty < h_\infty < \infty$, since that $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [g_\infty, h_\infty]) \leq 1$, we show that $\lim_{t \rightarrow +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0$.

In fact, let $\bar{I}(x, t)$ denote the unique solution of the problem

$$\begin{cases} \bar{I}_t - d_I \bar{I}_{xx} + \alpha \bar{I}_x = \bar{I}(\beta(x, t) - \gamma(x, t)) - \frac{\beta(x, t)}{N^*} \bar{I}^2, & g_\infty < x < h_\infty, t > 0, \\ \bar{I}(g_\infty, t) = 0, \quad \bar{I}(h_\infty, t) = 0, & t > 0, \\ \bar{I}(x, 0) = \tilde{I}_0(x), & g_\infty < x < h_\infty, \end{cases} \quad (4.2)$$

with

$$\tilde{I}_0(x) = \begin{cases} I_0(x), & g_0 \leq x \leq h_0, \\ 0, & \text{otherwise.} \end{cases}$$

The comparison principle gives $0 \leq I(x, t) \leq \bar{I}(x, t)$ for $x \in [g(t), h(t)]$ and $t \geq 0$.

Using the fact $R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [g_\infty, h_\infty]) \leq 1$, we know that (Lemma 3.1 (ii)) the principal eigenvalue λ_0 of (3.3) is nonnegative. the corresponding T-periodic problem

$$\begin{cases} U_t - d_I U_{xx} + \alpha U_x = U(\beta(x, t) - \gamma(x, t)) - \frac{\beta(x, t)}{N^*} U^2, & x \in (g_\infty, h_\infty), t > 0, \\ U(g_\infty, t) = U(h_\infty, t) = 0, & t > 0, \\ U(x, 0) = U(x, T), & x \in [g_\infty, h_\infty]. \end{cases} \quad (4.3)$$

admits only trivial solution 0. It is shown in [8, 14], by the method of upper and lower solutions and its associated monotone iterations, that the time-dependent solution $\bar{I}(x, t)$ converges to 0 uniformly in $[g_\infty, h_\infty]$ as $t \rightarrow \infty$. Therefore, $\lim_{t \rightarrow +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0$.

In that case $-g_\infty = \infty = h_\infty$, we first consider problem (4.1). When $\alpha = 0$, the existence and uniqueness of positive T-periodic solution $U(x, t)$ of problem (4.1) is directed from Theorem 1.3 in [11]. When $\alpha \neq 0$, the result still holds since $\tilde{U} := N^*$ and $\hat{U} := \varepsilon \phi_\delta(x)$ are the ordered upper and lower solutions of problem (4.1), where ε and δ is a sufficiently small positive constant and ϕ_δ satisfies

$$\begin{cases} \phi_t - d_I \phi_{xx} + \alpha \phi_x = (\beta(x, t) - \gamma(x, t) - \delta) \phi, & x \in \mathbb{R}, t > 0, \\ \phi(x, 0) = \phi(x, T), & x \in \mathbb{R}. \end{cases} \quad (4.4)$$

By the assumption that (H_1) and (H_2) hold, problem (4.4) admits at one positive solution if $\delta < \frac{1}{T} \int_0^T (\beta_\infty(t) - \gamma_\infty(t)) dt - \frac{\alpha^2}{4d_I}$.

As to the limit, that is,

$$\lim_{n \rightarrow \infty} I(x, t + nT) = \hat{U}(x, t) \text{ locally uniformly for } (x, t) \in [0, \infty) \times [0, T],$$

the proof is based on the upper and lower solutions method, see Lemma 4.2 in [2] or Theorem 4.3 in [15]. \square

Now we give sufficient conditions so that the disease is spreading.

Theorem 4.2 *If $R_0^F(t_0) \geq 1$ for some $t_0 \geq 0$, then spreading must happen.*

Proof: Owing to $R_0^F(t_0) \geq 1$ for $t_0 \geq 0$, $R_0^F(t_1) \geq 1$ for any $t_1 > t_0$ by the monotonicity in Theorem 3.4.

In this case, we have that the periodic-parabolic eigenvalue problem

$$\begin{cases} \psi_t - d_I \psi_{xx} + \alpha \phi_x = \beta(x, t) \psi - \gamma(x, t) \psi + \lambda_0 \psi, & x \in (g(t_1), h(t_1)), \\ \psi(g(t_1), t) = \psi(h(t_1), t) = 0, & t > 0, \\ \psi(x, 0) = \psi(x, T), & x \in [g(t_1), h(t_1)]. \end{cases} \quad (4.5)$$

admits a positive solution $\psi(x)$ with $\|\psi\|_{L^\infty} = 1$, where λ_0 is the principal eigenvalue. It follows from Lemma 3.1 that $\lambda_0 < 0$.

We are going to construct a suitable lower solutions to (1.2) and we define

$$\underline{I}(x, t) = \delta \psi(x, t), \quad g(t_1) \leq x \leq h(t_1), \quad t \geq t_1,$$

where δ is sufficiently small such that $0 < \delta \leq \frac{N^*(-\lambda_0)}{\bar{\beta}}$, $\delta \psi(x, t_1) \leq I(x, t_1)$ in $[g(t_1), h(t_1)]$ and $\bar{\beta} = \max_{[g(t_1), h(t_1)] \times [0, T]} \beta(x, t)$.

Direct computations yield

$$\begin{aligned} \underline{I}_t - d_I \underline{I}_{xx} + \alpha \underline{I}_x - (\beta(x, t) - \gamma(x, t)) \underline{I} + \frac{\beta(x, t)}{N^*} \underline{I}^2 \\ = \delta \psi(x) [\lambda_0 + \frac{\beta(x, t)}{N^*} \delta \psi(x)] \\ \leq 0 \end{aligned}$$

for all $t > 0$ and $g(t_1) < x < h(t_1)$. Then we have

$$\begin{cases} \underline{I}_t - d_I \underline{I}_{xx} + \alpha \underline{I}_x \leq (\beta(x, t) - \gamma(x, t)) \underline{I} - \frac{\beta(x, t)}{N^*} \underline{I}^2, & g(t_1) < x < h(t_1), \quad t > t_1, \\ \underline{I}(g(t_1), t) = \underline{I}(h(t_1), t) = 0, & t > t_1, \\ \underline{I}(x, t_1) \leq I(x, t_1)(x), & g(t_1) \leq x \leq h(t_1). \end{cases}$$

Using the comparison principle in the fixed interval $[g(t_1), h(t_1)]$ yields that $I(x, t) \geq \underline{I}(x, t)$ in $[g(t_1), h(t_1)] \times [t_1, \infty)$. It follows that $\liminf_{t \rightarrow +\infty} \|I(\cdot, t)\|_{C([g(t_1), h(t_1)])} \geq \delta \psi(0) > 0$ and therefore $h_\infty - g_\infty = +\infty$ by Lemma 4.1. \square

Similarly, we can also construct a suitable lower solution for I to obtain sufficient conditions so that the disease is spreading and construct some suitable upper solutions to derive sufficient conditions so that the disease is vanishing. The proof of the next theorem will be omitted since it is an analogue of Lemma 3.7 in [5] or Lemma 2.8 in [3].

Theorem 4.3 *Suppose $R_0^F(0) := R_0^D(d_I, \alpha, \beta(x, t), \gamma(x, t), [-h_0, h_0]) < 1$. Then spreading happens if μ is sufficiently large; and vanishing happens if $\|I_0(x)\|_{C([-h_0, h_0])}$ or μ is sufficiently small.*

The following result is a consequence of Corollary 2.3 and Theorem 4.3.

Theorem 4.4 (*Sharp threshold*) Fixed g_0, h_0 and I_0 . There exists $\mu^* \in [0, \infty)$ such that spreading happens when $\mu > \mu^*$, and vanishing happens when $0 < \mu \leq \mu^*$.

Finally we give the asymptotic spreading speeds when the spreading happens.

Theorem 4.5 Assume that (H_1) and (H_2) hold. If $h_\infty = -g_\infty = +\infty$, then

$$\lim_{t \rightarrow +\infty} \frac{h(t)}{t} = \frac{1}{T} \int_0^T k^*(\alpha, a, b)(t) dt, \quad \lim_{t \rightarrow +\infty} \frac{-g(t)}{t} = \frac{1}{T} \int_0^T k^*(-\alpha, a, b)(t) dt, \quad (4.6)$$

where $a(t) = \beta_\infty(t) - \gamma_\infty(t)$ and $b(t) = \frac{\beta_\infty(t)}{N^*}$, $(k, q) = (k^*(\alpha, a, b)(t), q^*(x, t))$ is the unique positive T -periodic solution of the problem

$$\begin{cases} q_t - d_I q'' + (k - \alpha)q' = q[a(t) - b(t)q], & x \in (0, \infty), t \in [0, T], \\ q(0, t) = 0, & t \in [0, T], \\ q(x, 0) = q(x, T), & x \in (0, \infty), \\ \mu q'(0, t) = k(t), & t \in [0, T], \end{cases} \quad (4.7)$$

Moreover,

$$0 < k^*(-\alpha, a, b) < k^*(0, a, b) < k^*(\alpha, a, b), \quad (4.8)$$

which means that the left boundary moves slower than the normal one without advection and the right boundary moves faster.

Proof: The existence and uniqueness of the solution (k^*, q^*) to problem (4.7) with $\alpha = 0$ is given by Theorem 2.4 in [4]. Checking the proof in [4], we found it still hold for $\alpha < 2\sqrt{d_I(\beta_\infty - \gamma_\infty)}$. Moreover,

$$0 < \overline{k^*}(\alpha, a, b) < 2\sqrt{d_I(\beta_\infty - \gamma_\infty)} + \alpha, \quad 0 < \overline{k^*}(-\alpha, a, b) < 2\sqrt{d_I(\beta_\infty - \gamma_\infty)} - \alpha,$$

where $\overline{\beta_\infty} = \frac{1}{T} \int_0^T \beta_\infty(t) dt$ and $\overline{k^*}(\alpha, a, b), \overline{\gamma_\infty}$ are defined similarly.

The proof of (4.6) is similar as that of Theorem 4.4 in [4] with obvious modification, see also Theorem 5.5 in [15] or Corollary 3 in [2].

(4.8) can be established as Proposition 1.2 in [7]. \square

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